Valuing a Constant-Growth Annuity: An Applied Approach Using a Financial Calculator

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This paper models the closed-form solutions of the present and future values of a constant-growth annuity in a manner that allows easy application of the time-value-of-money functions of a financial calculator. This model allows exact solutions and is valuable to practitioners and students for a number of applications that are often ignored in business classes, due to the inherent cumbersome mathematics. Example of constant-growth annuities include: retirement annuity contracts, insurance policies, leases, installment purchases, and court-awarded payments.

KEY WORDS: *Annuity, Valuation*

Introduction

An annuity is defined by Brigham (1992, p. 206) as a "series of equal payments made at fixed intervals for a specified number of periods". Webster's Dictionary (1992) provides a similar definition, but without requiring the payments to be of an equal amount. Brigham's definition provides the strict model requirements often necessary when teaching concepts to an introductory financial management class, however, for practical purposes, annuities often change in value. For example, many retirement annuity contracts stipulate growth in benefit payments to offset future inflation expectations. There are many other instances of financial arrangements with annuity payments where these payments increase based on some contractual rate or determined by some event. Examples include insurance policies, insurance benefits, lease contracts, installment contracts, and court-awarded payments.

Calculation of the value of a constant-growth annuity is often ignored when teaching financial mathematics and subsequently, is found confusing by many students and practitioners. The purpose of this paper is to demonstrate a simple approach for calculation of the value of a constant-growth annuity that allows use of a financial calculator. This paper develops the factors for the present value and future value of constant-growth annuities and demonstrates the proper adjusted interest rate, calculated as a function of the market and growth rates, and enables solution through use of a financial calculator.

Present Value of a Constant-Growth Annuity

To calculate the present value of a constant-growth annuity the common approach is to find the present value of a constant-growth perpetuity based on the first payment received, less the present value of the future constant-growth perpetuity cash stream that is not received. The formula to solve for the present value of a perpetuity can be represented as follows:

$$PV_0 = \frac{PMT_1}{k-g} \tag{1}$$

where PV_0 is present value at time 0 (today), PMT_1 represents the first annuity payment received at time 1 (one period in the future), k is the perceived market rate, and g the perceived growth rate.

Therefore, to calculate the present value of only the annuity payments actually received, we can subtract the present value of the future payments not received, from the present value of the entire perpetuity. The resulting equation represents the closed-form solution for the present value of an n-payment ordinary annuity that grows by a constant rate. An ordinary annuity is defined as an annuity where the first payment is received one period in the future.

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$$PV_0 = \left(\frac{PMT_1}{k-g} - \frac{PMT_n(1+g)}{k-g} \times \frac{1}{(1+k} \right. \quad (2)$$

In order to simplify the calculation we can restate this formula to make it fit the form of the **P**resent **V**alue **I**nterest **F**actor of **A**nnuity (PVIFA) Tables and, therefore, accommodate use of the Time-Value-of-Money (TVM) functions of financial calculators. First, the annuity payment is put in time 0 terms,

$$PV_0 = \left(\frac{PMT_0(1+g)}{k-g} - \frac{PMT_0(1+g)^n(1+g)}{(k-g)(1+k)^n} \right. \quad (3)$$

then terms are rearranged in equations (4) through (6).

$$PV_0 = \left(\frac{PMT_0(1+g)}{(1+k)-(1+g)} - \frac{PMT_0(1+g)}{((1+k)-(1+g))\frac{(1+k)^n}{(1+g)^n}} \right) \quad (4)$$

$$PV_0 = \left(\frac{PMT_0}{\left(\frac{1+k}{1+g}\right)-1} - \frac{PMT_0}{\left(\left(\frac{1+k}{1+g}\right)-1\right)\left(\frac{1}{1}\right.} \quad (5)$$

$$PV_0 = \left(\frac{1}{\left(\frac{1+k}{1+g}\right)-1} - \frac{1}{\left(\left(\frac{1+k}{1+g}\right)-1\right)\left(1+\left(\frac{1+k}{1+g}\right)-1\right)^n} \right) PMT_0 \quad (6)$$

If we use the following identity

$$j = \left(\frac{1+k}{1+g} - 1 \right) \quad (7)$$

and substitute equation (7) into equation (6) we obtain,

$$PV_0 = \left(\frac{1}{j} - \frac{1}{j\,(1+j)^n} \right) PMT_0 \quad (8)$$

Since the $PVIFA_{1,n}$ at interest rate i for term n is stated as

$$PVIFA_{i,n} = \left(\frac{1}{i} - \frac{1}{i\,(1+i)^n} \right) PMT_1 \quad (9)$$

equation (8) tells us that the present value of a constant-growth ordinary annuity can be solved using the PVIFA for term n at an equivalent interest rate of

$$i = \frac{1+k}{1+g} - 1 \quad (10)$$

calculated using discount rate k and growth rate g. However, note that although the first payment of an ordinary annuity is not received for one period, we must treat the annuity payment in time 0 dollars [see equation 8].

Application: Present Value of a Constant-Growth Annuity

As an example, assume we want to value a retirement annuity contract that is offered through an insurance company. If this contract stipulates 20 annual payments, the first payment is in the amount of $52,500 payable one year from today, and payments increase at a rate of 5% per year. Assuming a discount rate of 7% we can value this annuity contract using the equation (2) formula or the TVM functions of a financial calculator based on equation (8). Using the equation (2) formula, the present value of this annuity contract is determined as follows [PMT_1=$52,500, g=5%, k=7%, n=20]:

$$PV = \left(\frac{PMT_1}{k-g} - \frac{PMT_n(1+g)}{k-g} \times \frac{1}{(1+k)^n} \right) \quad (11)$$

$$PV = \left(\frac{52500}{.07-.05} - \frac{\left(50,000(1+.05)^{20}\right)(1+.05)}{.07-.05} \times \frac{1}{(1+.07)^{20}} \right) = \$825,135.81 \quad (12)$$

In an equivalent manner we may use the TVM functions of a financial calculator to solve this constant-growth ordinary annuity at an interest rate equal to

$$i = j = \frac{1+k}{1+g} - 1 = \frac{1+.07}{1+.05} - 1 = 1.904762\% \quad (13)$$

and by using a time 0 equivalent payment of \$50,000 [n=20, PMT_0=\$50,000, i=1.904762%]. Based on these calculator inputs, we find a present value [PV = \$825,135.81] equivalent to the formula answer given in equation (12).

Therefore, as demonstrated, solving for the present value of a constant-growth ordinary annuity can be achieved using the TVM functions of a financial calculator after making two adjustment. First, we are required to make a discount rate adjustment as shown in equations (10) and (13). Second, we must adjust our payment to the equivalent time 0 payment.

Similarly, we may solve for the present value of a constant-growth annuity due by using the same discount rate shown in equations (10) and (13), but with the time 0 payment compounded one period at the rate k [PMT_0(1+k) = \$50,000(1.07) = \$53,500].[2] Based on use of a financial calculator the solution is \$882,895.31. We define a constant-growth annuity due as a series of payments made at fixed intervals, growing at a constant rate, where the first payment is received today.

In addition, the present value of an annuity due can be obtained by solving for the present value of an ordinary annuity and then compounding the value one period at the rate k [PV(1+k) = \$825,135.81(1.07) = \$882,895.31].

Future Value of a Constant-Growth Annuity

In a similar manner, we can demonstrate that the future value of a constant-growth annuity can be solved for term n at an interest rate equivalent to

$$i = \frac{1+k}{1+g} - 1 \quad (14)$$

based on a discount rate k and growth rate g, with the time 1 payment adjusted by $(1+g)^{n-1}$.

The closed-form equation of the future value of a n-period ordinary annuity growing at a constant rate can be represented as

$$FV_n = \left(\frac{PMT_1}{k-g} \times (1+k)^n - \frac{PMT_n(1+g)}{k-g} \right) \quad (15)$$

which can be rewritten as

$$FV_n = \left(\frac{\left(1+\left(\frac{1+k}{1+g}\right) - 1\right)^n - 1}{\left(\frac{1+k}{1+g}\right) - 1} \right) PMT_0 \, (1+ \quad (16)$$

Again, if we use the identity

$$j = \left(\frac{1+k}{1+g} - 1 \right) \quad (17)$$

and by substituting equation (17) into equation (16) we obtain

$$FV_n = \left(\frac{(1+j)^n - 1}{j} \right) PMT_1 \, (1+g)^{n-1} \quad (18)$$

Since the **F**uture **V**alue **I**nterest **F**actor of an **A**nnuity at interest rate i for term n ($FVIFA_{i,n}$) is equal to

$$FVIFA_{i,n} = \left(\frac{(1+i)^n - 1}{i} \right) \quad (19)$$

equation (19) demonstrates that the future value of an n-period constant-growth ordinary annuity may be solved using the FVIFA at an equivalent interest rate of

$$i = j = \frac{1+k}{1+g} - 1 \quad (20)$$

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[2] *Note that the annuity due function of a financial calculator cannot be used since this function is based on a zero-growth annuity.*

and a time n equivalent payment [$PMT_1(1+g)^{n-1} = PMT_n$].

Application: Future Value of a Constant-Growth Annuity

Based on the previous example of a 20-year ordinary annuity that grows at 5% per year, with the first payment of $52,500 paid in year 1, and subject to a discount rate of 7%, we may solve for the future value (20 years into the future) based on equation (15);

$$FV = \left(\frac{PMT_1}{k-g} \times (1+k)^n - \frac{PMT_n(1+g)}{k-g} \right) \quad (21)$$

$$FV = \left(\frac{50000 \times (1.05)}{.07-.05} \times (1.07)^{20} - \frac{50000(1.05)^{20}(1.05)}{.07-.05} \right) \quad (22)$$

$$= \$3{,}193{,}015.23$$

Similarly, we may solve for the future value of this constant-growth ordinary annuity using the time-value-of-money functions of a financial calculator after making the same interest rate adjustment shown in equations (10), (13), and (20)

$$i = j = \frac{1+k}{1+g} - 1 = \frac{1+.07}{1+.05} - 1 = 1.904762 \quad (23)$$

and setting payment equal to $132,664.88 ($PMT_1(1+g)^{n-1} = \$52{,}500(1.05)^{19} = \$132{,}664.88$). The future value solution based on these parameters is $3,193,015.27.

To solve for the future value of a constant-growth annuity due we discount the payment by the growth rate for one period so that payment becomes $PMT_{20}(1+k)$ [where $PMT_{20} = PMT_1(1+g)^{n-1}$]. Based on use of a financial calculator the solution is $3,416,526.34.

Alternatively, the future value of a constant-growth annuity due can be valued by compounding the future value of an ordinary annuity one period at the rate k [FV(1+k) = $3,193,015.27(1.07) = $3,416,526.34].

Conclusions

This paper models the closed-form solutions of the present and future values of a constant-growth annuity in a manner that allows easy application of the time-value-of-money functions of a financial calculator. Through this model, practitioners and students may easily solve for the value of various constant-growth annuities, such as, retirement annuity contracts, insurance policies, lease obligations, installment contracts, and court-awarded annuities.

References

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