Constant Mix Portfolios and Risk Aversion

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Merton (1969) established the conditions under which a constant mix portfolio strategy is optimal across a multi-period investment horizon. A central tenant of Merton's paper is that the investor's portfolio must be rebalanced continuously. In practice, portfolios are not rebalanced continuously, especially by individual investors. As portfolios are rebalanced less frequently the asset mix will drift from the target weights of the constant mix strategy and lower investor utility. We measure this 'drift' indirectly by measuring the difference between the investor's level of risk aversion and the risk aversion that would make the non-rebalanced portfolio an optimal choice.
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Introduction

Merton (1969) showed the conditions under which constant mix portfolio strategies are optimal. If risky asset prices follow geometric Brownian motion, there exists a risk-free asset paying a constant rate of interest, and portfolios can be rebalanced continuously without cost, then investors exhibiting constant relative risk aversion will find it optimal to maintain constant allocations between the risky and risk-free asset, with investment in the risky asset inversely related to the investors level of risk aversion.

As a practical matter, the existence of transactions fees, taxes, and other explicit and implicit costs make continuous rebalancing economically infeasible. As the rebalancing interval is increased it is more likely that the mix of assets in the portfolio will drift from the intended constant weights suggested by Merton's (1969) model. If the portfolio becomes over-weighted (under-weighted) in the risky asset, the level of risk aversion implied by the un-rebalanced portfolio will be lower (higher) than the investor's actual level of risk aversion. That is, higher (lower) allocations to the risky asset would be optimal for investors with lower (higher) risk aversion. For the investor whose portfolio has drifted away from its desired mix, the new portfolio mix may or may not lead to a better risk-return trade-off. However, without rebalancing this portfolio will be sub-optimal for the investor, resulting in a loss of utility.

Figure 1 illustrates the impact that discrete rebalancing can have upon constant mix portfolios. In a practical application of utility theory, financial counselors are interested in measuring an investor's level of risk aversion (Droms and Strauss, 2003; Cordell, 2004). They determine the maximum level of risk consistent with the client's risk aversion, and then select the appropriate portfolio along the capital market line (CML) by allocating funds between a risk-free bond, r, and a risky index fund, M. For instance, an investor who is moderately risk-averse might choose portfolio P, while investors with lower and higher risk aversion might choose portfolios S and Q, respectively. If the investor holding portfolio P rebalances infrequently, then the mix of risky to risk-free asset will change over time. For example, in the event of a bull market the portfolio might drift toward a higher weight in the risky asset indicated by portfolio S. While portfolio S is optimal for an investor with a lower level of risk aversion, it is sub-optimal for the moderately risk-averse investor, relegating him to a lower indifference curve, U'_2, with a loss of utility. By rebalancing back to his original portfolio allocations this investor can re-establish himself on indifference curve U'_1, thereby maximizing his utility. Conversely, in a bear market the portfolio might drift towards a relatively higher weight in the risk-free bond and place the investor at point Q. This portfolio is optimal for an investor with a relatively higher degree of risk aversion, but for the moderately risk-averse investor the portfolio is sub-optimal, lying on a lower indifference curve, U'_2. Rebalancing allows the investor to re-establish himself at point P along the CML.

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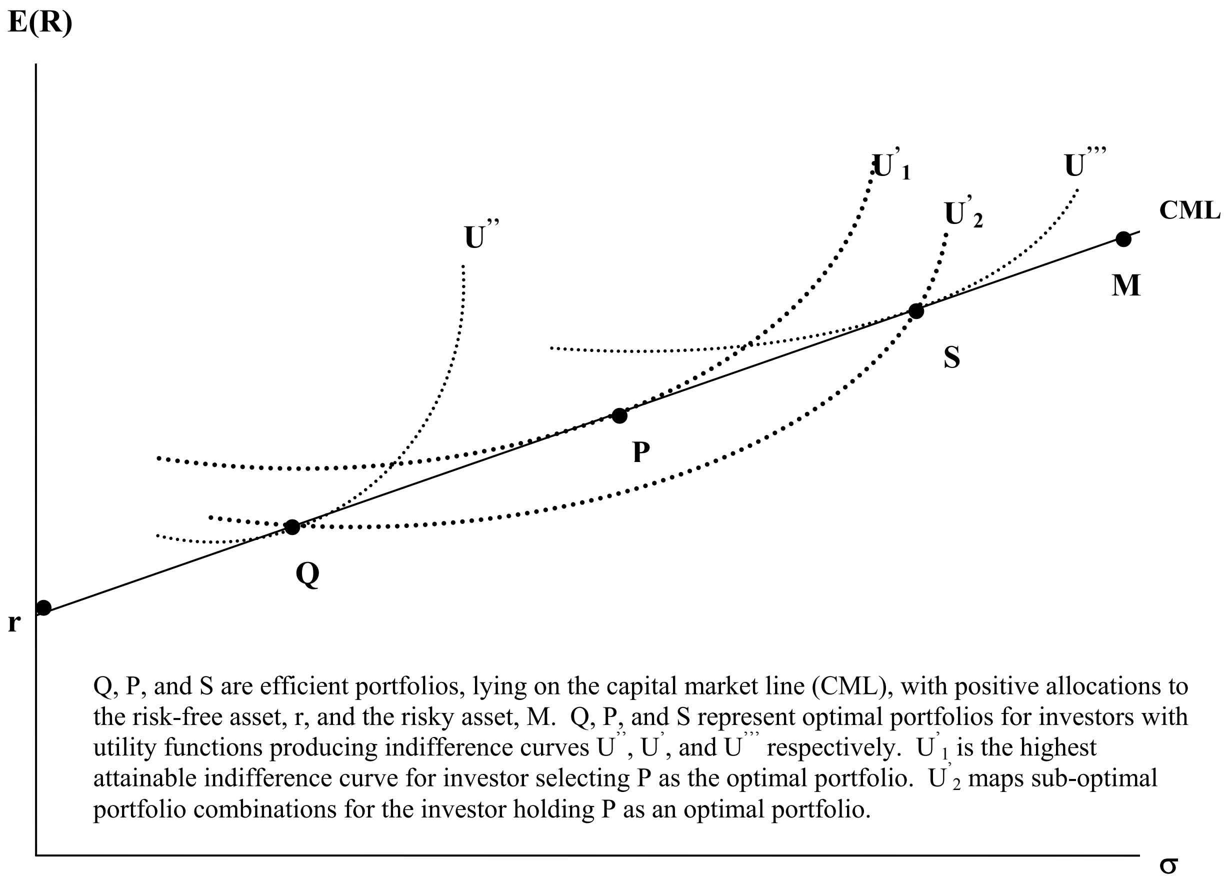
By measuring the difference between the investors risk aversion and the implied risk aversion of his portfolio prior to rebalancing we measure the extent to which an infrequently rebalanced portfolio will be inconsistent with the investor’s preferences. Since financial counselors are more likely to think of investor preferences in terms of investor risk aversion rather than investor utility, we think that this approach is beneficial from a practical standpoint. If the ‘implied’ risk aversion prior to rebalancing varies significantly from the investor’s actual level of risk aversion, then it is an implicit signal that the investor should rebalance his portfolio more frequently. As such, we use Monte Carlo simulation to address two questions: (1) How sensitive is the level of implied risk aversion to the selection of rebalancing interval, and (2) are investors with different levels of risk aversion equally sensitive to the selection of rebalancing interval? Our model is presented in section two, with results in section three, and conclusions in section four.

Figure 1. Graphical Analysis of Portfolio Drift and Risk Aversion



Q, P, and S are efficient portfolios, lying on the capital market line (CML), with positive allocations to the risk-free asset, r, and the risky asset, M. Q, P, and S represent optimal portfolios for investors with utility functions producing indifference curves U'', U', and U''' respectively. U'_1 is the highest attainable indifference curve for investor selecting P as the optimal portfolio. U'_2 maps sub-optimal portfolio combinations for the investor holding P as an optimal portfolio.

Model

Monte Carlo simulation is used to analyze an investor who implements a constant mix strategy by allocating wealth between two assets, a risk-free bond and a risky market index fund at discrete intervals and over a finite horizon.

Our simulation is designed to be consistent with capital asset pricing theory and two-fund separation such that investors hold efficient portfolios along the capital market line (CML) by allocating wealth between a risk-free bond and the risky market portfolio. Our risky asset is assumed to be one that mimics the market portfolio. In practice, this would be accomplished by investing some amount of wealth in a stock index mutual fund. This is consistent with evidence provided by Hariharan, Chapman, and Domian (2000) who studied investors in the 51–61 age bracket and found that they allocated relatively less wealth to risky assets as their relative risk aversion increased. Further, they showed that investors tended not to alter the construction of the risky part of their portfolio.
The risk-free bond grows at a constant, continuously compounded rate, r:

(1) $$B_{t+1} = B_t e^{r\Delta t}$$

The risky market index fund grows according to a discrete version of a standard diffusion process:

(2) $$\Delta M_t / M_t = \mu \Delta t + \sigma \tilde{\varepsilon} \sqrt{\Delta t}$$

where μ and σ are the continuously compounded mean and standard deviation, and $\varepsilon \sim$ N(0,1).

Investor preferences are assumed to be characterized by constant relative risk aversion (CRRA) such that investors are averse to a given percentage loss in wealth. Thus, they will hold risky and risk-free assets in fixed proportions that will be independent of their level of wealth. Consistent with this assumption, we use a power utility function with coefficient of relative risk aversion, a, defined over terminal wealth, W_T.

(3) $$U\left(\tilde{W}_T\right) = \frac{1}{1-a}\tilde{W}_T^{1-a}$$

Following Merton (1969), given a constant risk-free rate, a risky asset that evolves according to geometric brownian motion, CRRA, and continuous, costless rebalancing, the optimum allocation to the risky asset is constant for all periods, t:

(4) $$x_t = \frac{\mu - r}{\sigma^2 a} = x \quad \forall \quad t$$

In our simulation the investor rebalances over a discrete interval, τ. At the end of each interval, prior to rebalancing, the investor's wealth is computed as follows:

(5) $$\begin{aligned} W_{t+\tau} &= B_{t+\tau} + M_{t+\tau} \\ &= W_t\left[(1-x)e^{r\tau} + x\left(M_{t+\tau}/M_t\right)\right] \end{aligned}$$

With discrete rebalancing the actual allocation to the risky asset will vary between rebalancing dates. In turn, this implies a level of risk aversion different from the investor's actual or true level of risk aversion. The implied risk aversion, a^I, is determined by solving equation (4) with x equal to the actual holding of the risky asset to total wealth just before the portfolio is rebalanced back to the investor's intended constant weights.

(6) $$a^I_{t+\tau} = \frac{\mu - r}{\sigma^2 \, M_{t+\tau} / W_{t+\tau}}$$

The implied risk aversion coefficient is computed just before each rebalancing date in the investor's horizon, and in turn these values are used to compute the average and standard deviation of a^I over the investment horizon. The average implied risk aversion across the investment horizon is given by

(7) $$\bar{a}^I_j = \frac{1}{t}\sum_{t=1}^{\psi} a^I_{t+\tau}$$

where ψ is the number of rebalancing intervals per investment horizon. The standard deviation of the implied risk aversion across the investment horizon is given by

(8) $$\sigma_{a^I_j} = \sqrt{\frac{1}{t}\sum_{t=1}^{\psi}\left(a^I_{t+\tau} - \bar{a}^I\right)^2}$$

The two statistics computed from equations (7) and (8) measure the mean and standard deviation of the implied level of risk aversion across a single trial, j, in the simulation. This process is repeated for each trial in our simulation. At the end of the simulation we compute the averages of $\bar{a}^I_j$ and $\sigma_{a^I_j}$ as follows:

(9)
$$E\left(a^I\right)=\frac{1}{j}\sum_{j=1}^{1000}\bar{a}_j^I$$

(10)
$$V\left(a^I\right)=\frac{1}{j}\sum_{j=1}^{1000}\sigma_{a_j^I}$$

where E(a^I) and V(a^I) are the expected value and volatility of the implied level of risk aversion, respectively.

The implied and true levels of risk aversion differ from each other as a result of drift in the underlying portfolio weight. Consequently, we report in Table 1 the actual portfolio weight just prior to rebalancing. For each trial we compute the average portfolio weight before rebalancing, $\bar{x}_j$, using equation (11). At the end of the simulation we average these values across all trials in the simulation by using equation (12). This procedure is repeated for each rebalancing interval and initial portfolio weight.

(11)
$$\bar{x}_j=\frac{1}{t}\sum_{t=1}^{\Psi}x_{t+\tau}$$

(12)
$$E\left(x\right)=\frac{1}{j}\sum_{j=1}^{1000}\bar{x}_j$$

For all simulations the risk-free bond is assumed to grow at a constant rate, r = 6.8%. The risky asset is assumed to have a constant mean, μ, of 14.1% and standard deviation, σ, of 15.2%. Parameter values for μ, σ, and *r* are taken from Jensen and Mercer (2003), and are based on their estimates of the annual return and standard deviation on large company stocks and the return on Treasury bills for the period 1972-1999.

A transaction fee of 0.5% is taken against the value of the portfolio rebalanced each period (Do, 2002). Simulations were also run without transactions costs. However, because the results in Table 1 arise from relative as opposed to absolute changes in portfolio weights, they are insensitive to changes in transactions costs.

Lastly, the simulation is conducted across 1,000 trials, where each trial in the simulation assumes a 10 year investment holding period.

Table 1.
Implied Risk Aversion for Non-Continuous Rebalancing

	x = 25%			*x* = 50%			*x* = 75%		
Rebalancing Interval	E(a^I)	V(a^I)	E(x)	E(a^I)	V(a^I)	E(x)	E(a^I)	V(a^I)	E(x)
Continuous	12.64	0.03	25.00	6.32	0.01	50.00	4.21	0.00	75.00
Weekly	12.63	0.20	25.03	6.32	0.07	50.03	4.21	0.02	75.02
Monthly	12.60	0.41	25.11	6.31	0.14	50.13	4.21	0.05	75.09
Quarterly	12.52	0.70	25.32	6.28	0.23	50.39	4.20	0.08	75.26
Semi-Annual	12.40	0.96	25.64	6.24	0.32	50.78	4.19	0.11	75.53
Annual	12.17	1.26	26.29	6.16	0.42	51.55	4.16	0.14	76.04
Bi-Annual	11.72	1.54	27.61	6.01	0.51	53.08	4.11	0.17	77.05

Results and Implications for Financial Planners

Simulation estimates of the implied level of risk aversion, the volatility of this value, and the drift in the risky asset allocation are presented in Table 1. Simulations are conducted for risky allocations of 25%, 50%, and 75% and rebalancing intervals ranging from continuous to bi-annual. The reported values of E(a^I), V(a^I), and E(x) are computed just prior to rebalancing according to equations (9), (10), and (12), respectively. All three of these statistics vary with the rebalancing interval. Reading down each column shows that the longer the period before rebalancing of a portfolio back to its intended constant mix, the greater the relative amount of drift in the portfolio and consequently, the greater the drift and volatility in the implied level of risk aversion associated with the portfolio. Reading across Table 1 shows that drift in risk aversion and its volatility are inversely related to the proportion of wealth allocated to the risky asset.

These results have important practical implications for financial planners. Financial counselors often advise their clients to reduce their relative holdings of risky assets and move the money into less volatile securities as the client ages. This concept, referred to as life cycle investing, proposes that investor risk aversion increases with age. Consequently, investors should reduce portfolio weighting to risky assets.

Given the assumptions of Merton's (1969) model, an investor will optimally maintain the same constant portfolio weights over time. Thus, as the investor ages, he will still hold the same portfolio. Though Merton's (1969) paper opened the way for research into dynamic portfolio modeling, it doesn't fit well with the practicalities of life cycle investing. In recent years researchers have modified Merton's (1969) approach by adding labor income (Bodie, Merton, and Samuelson, 1992), transactions costs (Liu and Loewstein, 2002), and stochastic bond rates (Bajeaux-Besnainou, Jordan, and Portait, 2003) to develop models that are consistent with the life cycle investment advice so often proposed by financial advisors.

A key element of life cycle theory is the idea that investors should decrease their allocation to risky assets as they age due to an increase in their level of relative risk aversion. While investor age is a central element of life cycle theory, much of the research into investor risk aversion has considered many factors that are thought to influence investor utility, age being one of them. A common approach is to infer the level of an investor's relative risk aversion by observing how the investor's allocation of wealth to risky assets changes in response to changes in these factors. This approach is taken by Morin and Suarez (1983) and Bakshi and Chen (1994) who test the life cycle hypothesis by regressing observed changes in allocations of wealth to risky assets onto several independent variables, including age. Both studies find support for the life cycle hypothesis by finding that allocations to risky assets decrease as investors grow older. In a different approach, Sung and Hanna (1996) take survey data that asks investors to define their level of risk tolerance -- the inverse of risk aversion -- and find that risk tolerance decreases as investors draw nearer to retirement. However, they do not find that age is significantly related to risk tolerance. Riley and Chow (1992) and Wang and Hanna (1997) also study observed changes in risky asset allocations relative to wealth to make inferences about investor relative risk aversion. They find that a greater proportion of wealth is allocated to risky assets as one ages, but abruptly decreases once the investor reaches retirement. The results of Riley and Chow (1992), Sung and Hanna (1996), and Wang and Hanna (1997) suggest that relative risk aversion decreases with age, but increases due to retirement. As a whole, this body of literature suggests that an investor's relative risk aversion eventually decreases, whether gradually as one approaches retirement, or abruptly upon retirement. Thus, while it is difficult to say that age has a consistent impact on changes in risk aversion, the evidence is more conclusive that risk aversion does tend to decrease the closer one draws to retirement. For an extensive review of the literature on risk aversion measurement see Hanna, Gutter, and Fan (2001).

While far from conclusive, this volume of research suggests that a relatively young investor should allocate a high proportion of wealth to risky assets, and that the investor should substitute the risk-free bond for the risky fund as he approaches retirement and becomes relatively more risk-averse. As our results show, the greater the proportion of wealth in the risk-free bond, the more sensitive the investor should be to the selection of rebalancing interval. The volatility numbers we provide in Table 1 are useful in this regard. The volatility of the implied risk aversion coefficient increases as the allocation to the risky asset decreases. That is, the more risk-averse the investor, and consequently the more the investor allocates to the risk-free asset, the more likely the portfolio will drift relative to its intended target constant weights. This can be seen by observing that the values for E(x) in Table 1, measuring the change in portfolio weight just prior to rebalancing, increase by approximately the same amount for each portfolio as the rebalancing frequency decreases. However, the change in portfolio weight is greater relative to the initial or intended weighting for investors with a lower allocation to the risky asset. This effect is also picked up by the volatility of the implied level of risk aversion. Greater volatility in the implied level of risk aversion occurs as a result of greater volatility in the investor's portfolio allocations prior to rebalancing. For example, our simulations provide a volatility of 0.96 for an investor allocating 25% to the risky asset and rebalancing on a semi-annual basis, and only 0.11 for an investor allocating 75% to the risky asset while also rebalancing semi-annually. This suggests that the more risk-averse of the two investors will be exposed to a greater chance of the portfolio allocations drifting from the target weight of 25% in the risky asset.

Though the results are not reported here, we ran simulations tied to a range of parameter values, and while the values for $E(a^I)$ and $V(a^I)$ changed as a result, the main findings of this study were preserved. That is, in all simulations, the drift in $E(a^I)$ and its associated volatility, $V(a^I)$, increased as the rebalancing interval increased, and the amount and volatility of this drift were inversely related to the allocation in the risky asset.

While age and looming retirement can alter an investor's relative risk aversion, so do other factors. For example, Bajtelsmit, Bernasek, and Jianakoplos (1999) show that women tend to exhibit higher risk aversion than men, all else equal; while Riley and Chow (1992) show that risk aversion tends to change inversely with wealth, education, and income. Thus, given our results a financial planner might be inclined to advise women to rebalance more frequently than men, while they might advise wealthier clients or clients with greater current income to rebalance less frequently.

Summary and Conclusions

The less frequently investors revise their portfolios the more likely their portfolio will drift relative to their target weight and desired level of risk aversion. The degree to which this occurs is a positive function of the proportion of the portfolio allocated to the risk-free asset. That is, while the change in the portfolio weight is roughly the same for each portfolio, this drift constitutes a greater change as a percentage of the intended weight. This suggests that the more risk averse the investor, the greater the sensitivity of the investor to a given change in his portfolio weight. This result is consistent with the theoretical results of Hawawini (1986) who showed that the sensitivity of investors to changes in their portfolio can be measured by the curvature of their indifference curves. The rate at which the slope of an indifference curve changes given some change in risk (resulting from a change in portfolio weight) is equal to the investor's level of absolute risk aversion. For investors with a power utility function, absolute risk aversion is equal to the ratio of the relative risk aversion coefficient divided by wealth. Thus, the more risk averse the investor (i.e. the greater the value for the relative risk aversion coefficient, a), the more sensitive the investor will be to a given change in his underlying portfolio. Tying together our results with those of Hawawini (1986) suggests that relatively more risk-averse investors are more sensitive to changes in their portfolio, and as a result they should rebalance their holdings more frequently than would less risk-averse investors.

Financial counselors are already aware that more risk-averse individuals should lower their allocation to risky assets. Our research suggests that such investors should be advised to rebalance their portfolios more frequently than would investors with relatively higher allocations to the risky asset. As other research has shown, groups that tend to have higher relative risk aversion and therefore lower risky allocations include investors nearing or in retirement, women, investors with low levels of wealth, both financial and real, low income investors, and investors with less financial education.

Portfolio theory tells us that as investor risk aversion increases investors will sacrifice greater amounts of expected return in order to lower risk by a given amount. Consequently, a given amount of drift in the implied level of risk aversion would be more troubling to an investor with a high degree of risk aversion. Thus, an extension of our research would be to examine the actual sensitivity of investors to changes in relative risk aversion and the resultant changes in allocation to the risky asset to determine if highly risk-averse investors are truly more sensitive to a given change in their portfolio than are relatively less risk-averse investors.

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