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The Rule Of 72

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*The **Rule of** 72, a staple in financial circles for estimating the amount of time required for an investment to double in value, is shown to be quite inaccurate at today's high rates of return. The derivation of the **Rule of** 72 provides insight into how the rule can be improved. A new rule of thumb, which provides a very high level of accuracy, is obtained by a simple regression.*
Keywords: *Future value, Rule of 72, Regression, Retirement wealth, Saving*

Introduction

The *Rule of 72* is widely quoted in the popular press and by financial advisors as a quick and reasonably accurate way of measuring the amount of time, in years, needed to double an investment at an expected rate of return of r * 100% per annum. The rule is also used to express the required rate of return for a given time period that would double an investment with re-investment of all intermediate cash flows. The rule is simply stated as:

$$n = .72 / r \qquad (1)$$

where n is the number of years and r is the rate of return expressed as a proportion.

As it can be seen from Table 1, for interest rates around 6% and 12%, the *Rule of 72* is reasonably accurate. However, in the 1990s, annual rates of return on certain asset classes (read stocks) have been much higher than 12%. Under these circumstances the rule severely underestimates the amount of time required to double the value of an investment.

Here are some examples of the rule in action:

Example 1: For r = .12, the true n = 6.12 years. The *Rule of 72* estimates n to be 0.72 / 0.12 = 6.0, an underestimation of about 2%.

Example 2: For r = 0.36, the true n = is 2.25 years and the *Rule of 72* estimates it to be exactly 2 years, an underestimation of about 12%.

Ijiri (1972) showed that

$$r^* = i/(1 + (i/2)) \qquad (2)$$

where r* is an approximation of the actual nominal interest rate r and i is the effective interest rate equivalent of r under continuous compounding.

Using Ijiri's derivation, Gould and Weil[a] (1974) derived a significant improvement to the *Rule of 72*. They showed that :

$$t \cong 69/r^* + .35 \qquad (3)$$

where t is the time it takes for money earning r* percent per year to double in value. While (3) is a significant improvement over the *Rule of 72*, Gould and Weil (1974, p. 398) admitted that their "approximation becomes poorer, particularly at large values of r."

Table 1
True Amount of Time Required to Double an Investment at Selected Rates of Return, r, and Estimates of the Amount of Time Required by the *Rule of 72*

r	True n which doubles value	n Found by *Rule of 72*
0.06	11.90	12.00
0.12	6.12	6.00
0.18	4.19	4.00
0.24	3.22	3.00
0.30	2.64	2.40
0.36	2.25	2.00
0.42	1.98	1.71

In this paper we show that the Gould and Weil approximation can be further improved upon. First the *Rule of 72* will be derived; from this derivation a better estimation formula (*rule*) can be derived. Second, a regression model will be estimated, which will also suggest an alternative *rule*. The reader may choose which rule to use; either will be an improvement over the existing rule of 72 for large r.

Deriving of the Rule of 72

We know that:

$$F = P(1+r)^n \qquad (4)$$

where F is Future Value and P is Present Value.

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If the Future (F) is twice P, then (4) may be rewritten as:
$$2 = (1+r)^n \qquad (5)$$
and taking the natural log of both sides results in
$$\ln 2 \approx 0.693 = n \ln(1+r). \qquad (6)$$
$\ln(1+r)$ may be expanded [for $-1 < r \leq 1$] by a Taylor's Series around zero (alternatively known as a Maclaurin Series) which results in
$$\ln(1+r) = r - 1/2\, r^2 + 1/3\, r^3 - 1/4\, r^4 + \ldots \qquad (7)$$
Truncating (7) after the first term gives the *Rule of 69*: .693 = nr, but this is not very accurate. Truncating after the second term gives .693 = nr(1-r/2) or
$$0.693/(1-r/2) \approx nr. \qquad (8)$$

For values of r between .02 and .15, the LEFT hand side of (8) varies between .70 .and. 75. For values of r around 0.05 to 0.12 the LEFT hand side of (8) varies between .71 and .73, so .72 is a pretty good approximation to the left hand side.

We can conclude that an approximation of the number of years, n, required to double an amount P at interest rates between 1 and 15% is approximated up to the second order [See (7)], by the *Rule of 72*: nr ≈ .72. (This is a pretty good approximation for rates between 6% and 12%.)

Stated another way, Equation (8) becomes
   0.693/(1-r/2) ≈ .72 = nr since 0.693/(1-r/2) is approximately .72 for .05 < r < .12.

Alternative Rule 1 - The Quadratic Rule
By using the second order approximation in (8) , a much more accurate, and not too complex improvement on the *Rule of 72* is the following:
$$n = .693/(r - r^2/2) \qquad (9)$$
Applying the rule to the examples above, the "new" results for the rule are n= 6.14 for example 1 and n=2.35 for example 2 (an overestimation of about 5%).

We note in passing that (9) will consistently overestimate n because it is missing terms in the denominator which would tend to make the denominator larger and therefore tend to make the result, n, smaller.

The *quadratic rule* is not easy to remember but it is possible to calculate with pencil and paper. It does not require either log tables or a calculator with log function. The next section shows a more accurate estimation rule which is simpler to remember.

Alternative Rule 2 - The Regression Rule
From Eq (6), we see that ln(2) = n ln(1+r) exactly and therefore n = ln(2)/ln(1+r) exactly. We know that n and r are inversely related; we attempt to estimate this relationship by specifying that
$$n = \alpha + \beta/r + \varepsilon \qquad (10)$$
where n is generated as $\ln(2)/\ln(1+r)$ for $0.01 \leq r \leq 0.50$, and ε is a stochastic term.

Using linear regression, we obtained the following estimates a and b, respectively, for α and β:

	n	=	a	+	b / r	(11)
			0.3315		0.6934 / r	
t-values			(372.39)		(14021.45)	

R^2=1.0   n=50   s=.005455
for values of r from 0.01 to 0.50 in increments of 0.01.

The newly derived rule is extremely accurate for rates of 1% to 50%. We call this rule the *Regression Rule*:
$$n = .693/r + .33 \qquad (12)$$
Trying this rule on example 1 gives n = 6.10, an underestimation of about 0.3 %. For example 2, we obtain n= 2.255 using Eq(12), an overestimation of about 0.2%.

Eq (12) is not much harder to remember than the *Rule of 72* in (1), but it provides much more accurate estimates for large r. A comparative evaluation of the robustness of the estimates obtained via the three rules is provided in Table 2.

Table 2
True Amount of Time Required to double an investment at selected rates of return, r, and estimates of the amount of time required from three Rules

r	True n	n Found by *Rule of 72*	n Found by Quadratic Rule	n Found by Regression Rule
0.06	11.90	12.00	11.91	11.88
0.12	6.12	6.00	6.14	6.10
0.18	4.19	4.00	4.23	4.18
0.24	3.22	3.00	3.28	3.22
0.30	2.64	2.40	2.72	2.64
0.36	2.25	2.00	2.35	2.26
0.42	1.98	1.71	2.09	1.98

As it can be seen from the table, the Regression Rule is more accurate for large r and small r too. The Rule of 72, for r = .02, overestimates the required time *by an entire year*. The new rule underestimates the required time *by about one week*.

Interestingly, Gould and Weil's (1974) rule is amazingly similar to the rule which we found by regression. While Gould and Weil's rule is a significant improvement to the *Rule of 72*, the rule derived by regression estimation is superior. In Table 3, we compare our regression results with Gould and Weil's results and with the exact answer. (The values of r in Table 3 are those used in the Gould and Weil paper).

Our results were closer to the exact results in 20 out of 21 cases. While Gould and Weil's Rule (a numerical approximation) and the Regression Rule (a statistical description) are strikingly similar, the Regression Rule has better performance.

Table 3
A Comparison of the True n, the *Regression Rule,* and the Gould/Weil Rule

r	True n	Regression Rule	Gould/Weil Rule of 69
0.00250	277.6053	277.5300	276.3500
0.00333	208.4987	208.4381	207.5572
0.00417	166.7017	166.6500	165.9500
0.00500	138.9757	138.9300	138.3500
0.00583	119.1715	119.1300	118.6357
0.00667	104.3193	104.2810	103.8510
0.00750	92.7658	92.7300	92.3500
0.01000	69.6607	69.6300	69.3500
0.01250	55.7976	55.7700	55.5500
0.01500	46.5555	46.5300	46.3500
0.01750	39.9540	39.9300	39.7786
0.02000	35.0028	34.9800	34.8500
0.03000	23.4498	23.4300	23.3500
0.06000	11.8957	11.8800	11.8500
0.08000	9.0065	8.9925	8.9750
0.10000	7.2725	7.2600	7.2500
0.12000	6.1163	6.1050	6.1000
0.20000	3.8018	3.7950	3.8000
0.40000	2.0600	2.0625	2.0750
0.50000	1.7095	1.7160	1.7300
1.00000	1.0000	1.0230	1.0400

Conclusion
We have shown that:
1. The approximation provided by the *Rule of 72* is a poor one for rates of return bigger than 15%, and
2. The "Regression Rule" provides highly accurate estimates of the amount of time required to double an investment.

The Regression Rule is not much harder to remember than the traditional one, and provides a significant increase in precision.

Endnotes
a.   We wish to thank an anonymous referee for pointing out the Gould and Weil article.

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